Bathymetry correction using an adjoint component of a coupled nearshore wave-circulation model

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The impact of assimilation of wave-averaged flow velocities on the bathymetric correction is studied in tests with synthetic (model-generated) data using tangent-linear and adjoint components of a one-way coupled nearshore wave-circulation model. Weakly and strongly nonlinear regimes are considered, featuring energetic unsteady along-beach flows responding to time-independent wave-averaged forcing due to breaking waves. It is found that assimilation of time-averaged velocities on a regular grid (mimicking an array of remotely-sensed data) provides sensible corrections to bathymetry. The adjoint-based representer formalism allows separating contributions of the circulation and wave adjoint model components to the bathymetric correction. In a test case involving a beach with an alongshore varying bar, the adjoint wave model contribution is to determine the cross-shore position of the bar crest. The adjoint circulation model provides an additional contribution, introducing alongshore variability in the shape of the bar. The array mode analysis reveals that there are very few modes that can be effectively corrected, given the assumed data error level. Bathymetry perturbations associated with these modes are a mixture of near-coast intensified modes as well as modes extending their influence to deep water (along the background wave characteristics). Additional tests show the utility of different observational arrays in providing the bathymetric correction.
1. Introduction

In the nearshore surf zone, knowledge of accurate bathymetry $h(x,y)$ is crucial in many applications, including wave and circulation forecasting and beach erosion monitoring. In response to storms, sand bars and other bathymetric irregularities can be formed, modified, or moved rapidly offshore, sometimes over a span of a day (Gallagher et al., 1998). Frequent high-resolution bathymetric surveys are difficult given harsh surf zone conditions as well as logistical constraints. Inversion of remotely-sensed observations of oceanic variables that are sensitive to bathymetry (including currents, wave set-up, celerity, dissipation, etc.) can provide an opportunity to obtain estimates of bathymetric changes without direct surveys. As an example, the Beach Wizard system (van Dongeren et al., 2008) utilizes estimates of wave celerity, wave dissipation, and intertidal bathymetry (derived from optical or radar observations of the nearshore ocean over an extended period of time) to correct bathymetry locally at points where data are available. The prior estimates of these variables are obtained from a numerical wave and circulation model, and a sequential assimilation scheme is devised to constrain bathymetry. This system can assimilate observations of variables that can be expressed in mathematical terms as analytical differentiable functions depending locally on $h$. Similarly, the c-Bathy algorithm (Holman and Plant, manuscript in preparation) provides estimates of bathymetry using wave celerity properties derived from the optical image analysis and local inversion of the wave dispersion relation.

Not every observed oceanic variable sensitive to $h$ can be written as a local analytical function of $h$. For instance, estimates of wave-averaged currents can be obtained from microwave radar imagery (Farquharson et al., 2005) or feature tracking using a sequence of
optical images (Chickadel et al., 2003). These currents are influenced by both local bathymetry and bathymetric variability at a distance. For example, bathymetry errors at some distance from the coast can affect estimates of shoreward wave propagation resulting in erroneous estimates of wave breaking, dissipation, and forcing of wave-averaged currents closer to the coast. In addition, bathymetry also enters explicitly into governing equations of the circulation and thus has a direct effect on a wave-averaged flow field. Hence, bathymetry estimation using observed nearshore currents requires a method that can account for both local and non-local dependence on $h$.

Some data assimilation methods (Bennett et al., 2002, Evensen, 2007) are designed to find ocean state estimates that fit the model and the data in a least squares sense. These methods use model error covariances (computed or specified $a$ $priori$) to provide interpolation and filtering of sparse data sets and to correct model inputs (including in our case bathymetry) at the locations where data are not available. The model error (and correction) is propagated in space and time using the model dynamics. Wilson et al. (2010) made a step in this direction assimilating in-situ velocity observations and correcting bathymetry, using an ensemble approach to compute model error covariances (Evensen, 2007). An ensemble of bathymetries was generated using empirical orthogonal functions (EOFs) of the bathymetry time series from a series of surveys in the study region. An ensemble of model solutions was obtained using this bathymetry ensemble. From these, the velocity-bathymetry error covariance matrix was obtained, which was used to form a transfer function between the velocity model-observation difference and the bathymetry correction. This method was also later applied in a river setting (Wilson and Özkan-Haller, 2012).
Variational methods (Bennett, 2002) compute error covariances in the model outputs given error covariances in the inputs. They are based on an explicit formulation of a cost function, which is a sum of quadratic terms penalizing deviations of the model inputs from their prior estimates and model-data differences. In practical applications, the minimum of the cost function is usually found iteratively, by running repeatedly a linearized version of the dynamical model (a so called tangent linear (TL) model) and its adjoint (AD) counterpart. Building tangent linear and adjoint models may be a challenging exercise. However, once these are developed, variational methods can provide advantages including: (i) a very clear and explicit formulation of statistical hypotheses about errors in model inputs (including bathymetry), (ii) ability to show error propagation in the model (in space and time) and (iii) ability to explore co-variability of model input and output errors. They also allow avoiding some of the problems of ensemble methods, including insufficient ensemble sizes, long-tail spurious correlations, and the filter degeneracy (Hamill et al., 2001). In the context of our study, the variational formalism will allow us to separate contributions of the wave and circulation models to the bathymetry correction.

Among the contributions utilizing the variational approach with nearshore models, Feddersen et al. (2004) tested the method with an alongshore-uniform surf zone model. They assimilated in-situ observations of pressure and velocities and included bottom friction as one of the corrected variables (but they did not correct $h$). Kurapov et al. (2007) developed tangent linear and adjoint components of a shallow-water nearshore circulation model and utilized these to demonstrate that variational assimilation can work over extended time windows in strongly nonlinear flow regimes, where the alongshore currents become unstable and eddies are shed. In that study, the dynamics and wave forcing formulation were borrowed from Slinn et al. (2000). The assimilation system was utilized with synthetic (model-generated) velocity data and provided
corrections to initial conditions and the circulation model forcing. Bathymetry or offshore wave
conditions were not included as control parameters and the TL and AD components of the wave
model were not developed. More recently, Veeramony et al. (2010) developed an adjoint
component of the nearshore spectral wave model SWAN. They assimilated observed two-
dimensional wave spectra to correct boundary conditions of the wave model. However, again, $h$
was not corrected.

In the present manuscript, we will explore opportunities that variational assimilation may
offer correcting $h$ in a coupled wave-circulation model. The adjoint component of the model,
which provides sensitivity to bathymetry, will be described in section 2. The model is similar to
that of Kurapov et al. (2007) and is idealized to only include one-way coupling. So, the incident
wave field is allowed to force nearshore currents; however, the resulting circulation does not
affect the incident wave field. We run a series of assimilation experiments with synthetic (model-
generated) data and demonstrate the impact of assimilation of velocities to correct errors in
bathymetry (section 3). Compared to other assimilation studies mentioned above, which
considered steady flows, we consider weakly and strongly nonlinear regimes, where alongshore
flows are unsteady (equilibrated waves or aperiodic eddy-shedding) in response to steady wave-
averaged forcing. Analyses of representer functions and array modes (section 4) are utilized to
help understand the relative contributions of the wave and circulation model components to the
bathymetry correction.

2. Model set-up.

Results discussed in this paper are obtained in a domain extending 256 m in the alongshore and
512 m in the offshore direction (Figure 1). The coastline is straight. Boundary conditions are
periodic in the alongshore direction. The model resolution is 2 m. A Cartesian coordinate system is introduced with \( x \) directed alongshore and \( y \) directed offshore. Figure 1a shows the true bathymetry in twin experiments, which is identical to that in Kurapov et al. (2007). It has the bar centered at \( y = 80 \) m. The bar height varies in the alongshore direction. The prior bathymetry (Figure 1b) has the bar displaced offshore by 20 m. Its shape and height is almost uniform in the alongshore direction (slight non-uniformity is allowed to excite alongshore variability and instabilities in currents). In twin experiments, we will run the nonlinear coupled model both with the true and prior bathymetries. The velocity values from the case with the true bathymetry will be assimilated in the model using the prior bathymetry and an attempt to correct \( h \) will be made.

The diagram in Figure 2a shows information flow in the nonlinear coupled model. Inputs to the wave model component (WAVE) include bathymetry \( h(x, y) \) and the incoming wave parameters in deep water (the latter are kept constant in our study and are not shown in the diagram). In all experiments below, similarly to Kurapov et al. (2007), we assume that narrow banded waves are incident at the offshore boundary at an angle of \( 135^\circ \), measured in the clockwise direction from the \( y \)-axis. The peak wave period is at \( T_p = 8 \) s. The root mean square (RMS) wave height in the deep water is \( H_{rms} = 0.7 \) m. The wave model (Slinn et al., 2000, Özkan-Haller and Li, 2003) yields fields of two components of the wave vector \( k \) and \( l \) (in the directions \( x \) and \( y \), respectively), wave energy \( E \), wave dissipation \( \epsilon \), and also forcing of the circulation model \( f = (f_x, f_y) \), which is obtained as the linear combination of derivatives of the components of the radiation stress tensor (see the Appendix for the wave model equations).

The nonlinear circulation model (denoted CIRC in Figure 2a) is based on the shallow-water equations with bi-harmonic horizontal dissipation and linear bottom drag (Kurapov et al., 2007):
\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (D\mu)}{\partial x} + \frac{\partial (D\nu)}{\partial y} = 0, \tag{1}
\]
\[
\frac{\partial (D\mu)}{\partial t} + \frac{\partial (D\mu u)}{\partial x} + \frac{\partial (D\nu u)}{\partial y} = -gD \frac{\partial \zeta}{\partial x} + f_x - ru - a\nabla^2 (h\nabla^2 u), \tag{2}
\]
\[
\frac{\partial (D\nu)}{\partial t} + \frac{\partial (D\mu v)}{\partial x} + \frac{\partial (D\nu v)}{\partial y} = -gD \frac{\partial \zeta}{\partial y} + f_y - rv - a\nabla^2 (h\nabla^2 v), \tag{3}
\]

where \( \zeta \) is the sea surface height, \( \mu \) and \( \nu \) are two orthogonal components of depth-averaged velocity, in the directions \( x \) and \( y \) respectively, \( D(x, y, t) = \zeta(x, y, t) + h(x, y) \), and \( a = 1.25 \) m\(^4\) s\(^{-1}\). Inputs to the circulation model (see Figure 2a) include forcing computed by the wave model \( f = (f_x, f_y) \), bathymetry \( h(x, y) \), and the vector of initial conditions \( \mathbf{u}(0) \) (which combines elements of \( \zeta \), \( \mu \), and \( \nu \)).

The response of the circulation model to the time-invariant \((f_x, f_y)\) depends on the value of the bottom drag coefficient \( r \). For large values, the system started from rest would equilibrate to a steady flow. For small \( r \), the alongshore current can become unstable. Two cases will be considered in this manuscript, similarly to Kurapov et al. (2007). In the first case \( r = 0.004 \) m s\(^{-1}\), and the flow regime is weakly nonlinear. After an approximately 1-hour spin-up period, the flow, started from rest, will achieve an equilibrated wave regime, with the currents meandering and flowing in the direction of negative \( x \). Several snapshots of velocity and vorticity \( \partial \mu / \partial x - \partial \nu / \partial y \) corresponding to this regime are shown in Figure 3 (top rows). In the second case \( r = 0.002 \) m s\(^{-1}\), and the flow regime is strongly nonlinear. After an approximately 1-hour spin-up period, the area-averaged kinetic energy of the system levels off and begins to vary near a certain level, but the solution is aperiodic, with spurious generation of energetic eddies (Figure 3, bottom rows).

The tangent linear (TL) model describes evolution of a small perturbation to the model inputs with respect to a given nonlinear model solution (background model state). The information flow
in the TL coupled model (Figure 2b) is similar to that in the nonlinear model. The linearized
coupled model has been built using wave and circulation model components (TL_W and TL_C).
The TL computer codes have been written by hand using recipes for line-by-line code
differentiation (Guiring and Kaminski, 1998). To build TL_C, we modified the shallow-water
model code used in Kurapov et al. (2007) to include perturbation in \( h \). This modification was
relatively easy since the original code already had dependency on perturbation of the total depth
\( D(x, y, t) = \zeta + h \). Specifically, in the original code, we had \( \delta D = \delta \zeta \), where \( \delta \) denotes the
perturbation variable. In the new code, \( \delta D = \delta \zeta + \delta h \), where the bathymetry perturbation \( \delta h \) is
provided as the input to the TL model.

The background solution is provided into TL_C (and its adjoint counterpart AD_C) as a
series of instantaneous fields from the nonlinear circulation model saved at specified time
intervals (every 30 s in the weakly nonlinear case and 15 s in the strongly nonlinear case). Then
the TL and AD circulation models determine background circulation fields at every time step by
linear interpolation between the saved snapshots.

Model TL_W had to be newly developed. To allow line-by-line code differentiation, the
original nonlinear wave model had to be modified in the following respect. Slinn et al. (2000)
solve wave propagation equations by integration along wave characteristics. The resulting \( k \) and \( l \)
are found along the wave characteristics and then interpolated on the regular Cartesian grid. To
avoid difficulties linearizing this two-dimensional interpolation algorithm, we replaced
integration along the wave characteristics by solving the following equation on the regular grid:

\[
\text{curl}(k, l) = \frac{\partial l}{\partial x} - \frac{\partial k}{\partial y} = 0 . \tag{4}
\]
This equation expresses conservation of wave crests (Dean and Dalrymple, 1991). It can be integrated numerically from the offshore boundary toward the coast, e.g., using a 4th order Runge-Kutta method, along lines $y = \text{const}$ of the original rectangular grid. In all other aspects (dispersion relation, wave energy equation, and parameterization of dissipation, see the Appendix) the wave model is easily differentiable.

Note that $\delta h$ influences the output of TL_C in two ways (see Fig. 1b). The first is direct, since $h$ enters equations (1)-(3) via $D$; the second is via the wave model TL_W, providing forcing due to the radiation stress divergence.

Since a discrete numerical model is utilized, a TL model implementation can be viewed as the result of matrix-vector multiplication, $\delta \alpha = [TL] \delta \phi$, where $\delta \phi$ and $\delta \alpha$ are the vectors of model inputs and outputs, respectively. As a part of code testing, we have verified that the result of TL_W implementation, $[TL_W] \delta \phi$, tends to the difference of two nonlinear models, $[NL_W](\phi + \delta \phi) - [NL_W]\phi$, as the magnitude of $\delta \phi$ is reduced.

Following the same matrix-vector interpretation, the adjoint (AD) model can be defined as the matrix transpose of $[TL]$: $[AD] = [TL]^T$. Of course, we do not store every element of $[TL]$ or its transpose. Instead, the AD model code provides the rule by which the matrix $[AD]$ multiplies an appropriate vector of inputs. It can be built by transposition of the TL code line-by-line (and reversing the order of operators) using recipes from Guiring and Kaminsky (1998). Both the circulation and wave components of the coupled AD model have been tested to guarantee that the matrix-product $Q^T [AD][TL]Q$ (where $Q = [\delta \phi_1 | \delta \phi_2 | \ldots | \delta \phi_n]$ is a collection of $n$ randomly generated TL model input vectors) is an $n \times n$ symmetric and positive-definite matrix (within machine precision).
In the scheme corresponding to the AD model (Figure 2c) the information flow is reversed compared to the TL model. In this scheme, symbols $\lambda$ with different superscripts denote various components of the input and output vectors. The set of the AD input fields has the same structure as the set of outputs of the TL model. In the practice of data assimilation, the AD model is forced by adding impulses to adjoint model variables at observation locations and times [for a more formal explanation, e.g., see (Kurapov et al., 2007, 2011)]. Observations of the wave properties (such as wave vector components) would provide impulse forcing of the wave model component $\text{AD}_W$ represented in Figure 2c by $\lambda^{(k)}$, and assimilation of those data types would not require the circulation model component $\text{AD}_C$. Data on currents assimilated in the model provide impulse forcing $\lambda^{(u)}(t)$ to $\text{AD}_C$, which in turn yield fields of sensitivity of the observed circulation variables to initial conditions $\lambda^{(u)}(0)$ and circulation model forcing $\lambda^{(f)}(t)$. The latter provides forcing of the $\text{AD}_W$ model component, in addition to $\lambda^{(k)}$. If wave variables are not assimilated, the $\text{AD}_W$ will only be forced by $\lambda^{(f)}(t)$. Note that $\text{AD}_C$, the adjoint of the time-stepping circulation model, is run backwards in time. Every time step, adjoint sensitivity to $h$ will be added to $\lambda^{(h,C)}$. Then, $\text{AD}_W$ will provide additional contribution $\lambda^{(h,W)}$ to yield the total sensitivity of flow velocities to bathymetry $\lambda^{(h)} = \lambda^{(h,C)} + \lambda^{(h,W)}$.

3. Assimilation experiments

The assimilation system discussed here can in general be used to assimilate any oceanic variables that can be matched to the model output. In particular, estimates of the wave vector components $(k, l)$, or wave celerity $(c_x, c_y) = (\omega/k, \omega/l)$, obtained from a sequence of radar or optical images, could be assimilated. The impact of these data types has been demonstrated, e.g., by Van Dongeren et al. (2008) and Holman and Plant (2012, manuscript in preparation) using methods...
based on local inversion. In our case, differences between the true and prior bathymetries are small enough such that the wave vectors are not substantially different. The relative difference in the cross-shore wave celerity components \( \frac{c_{y}^{\text{TRUE}} - c_{y}^{\text{PRIOR}}}{c_{y}^{\text{TRUE}}} \times 100\% \) is largest in the area of the bar, where it does not exceed 15\% (Figure 4). Assimilation of the wave celerity in a case like ours would be possible, but would require very accurate observations. For this reason, we do not include assimilation of the kinematic wave properties in our study and focus below on assimilation of wave-averaged flow velocities, which would allow us to understand details of how the adjoint component of the coupled model works.

Model currents appear to be appreciably sensitive to the bathymetry details in our case. For instance, Figure 5 shows cross-shore profiles of alongshore currents computed over the true and prior bathymetries, averaged in the alongshore direction and also averaged in time over the interval of (1, 2) h since the model was started from rest. In both weakly and strongly nonlinear cases, the largest differences are found in the area of the bar.

In the first series of assimilation tests, data sets are formed by sampling the true model velocities at points of a regular grid in an area extending 250 m offshore (Figure 6). The distance between neighboring data points is 10 m in both the alongshore and cross-shore directions. The variational data assimilation problem for a 2-hour spin-up case is considered. The true and prior models are both started from rest. Velocities from the true solution are averaged over the second half of the time interval, (1, 2) h, and sampled at the points of the observational array. Comparisons of the true and prior time-averaged velocity fields sampled at the observation locations show appreciable differences (Figure 7). Differences between the true and prior solutions turn out to be larger in the weakly nonlinear case than the strongly nonlinear case, which can be confirmed by plotting the magnitude of the difference of the time-averaged
currents \( \sqrt{(\vec{u}_{\text{TRUE}} - \vec{u}_{\text{PRIOR}})^2 + (\vec{v}_{\text{TRUE}} - \vec{v}_{\text{PRIOR}})^2} \), where \( \vec{u} \) and \( \vec{v} \) correspond to the velocity components averaged over interval \( t = (1, 2) \) h (Figure 8). The more energetic flow corresponding to a smaller bottom friction coefficient case appears to be less sensitive to the shape of the bottom.

Before assimilation, random noise is added to the time-averaged synthetic data, with the standard deviation of \( \sigma_d = 0.03 \) m s\(^{-1}\). The correction to \( h(x, y) \) is found by minimizing the cost function as follows:

\[
J = (h - h_{\text{PRIOR}})^T C_h^{-1}(h - h_{\text{PRIOR}}) + (d - Lu_{\text{PRIOR}})^T C_d^{-1}(d - Lu_{\text{PRIOR}}),
\]

subject to the exact model dynamics and exact initial conditions \( \zeta(0) = 0, u(0) = 0, v(0) = 0 \).

In (5), \( h \) is written as a vector including bathymetric values at all interior grid points; \( C_h(x_1, y_1, x, y) \) is the bathymetric error covariance matrix, with elements depending on the location of grid points \((x_1, y_1)\) and \((x, y)\); \( d \) is the vector of all time-averaged observations, \( u_{\text{PRIOR}} \) is the prior model solution; \( L \) is the operator matching the model solution and the observations (which involves sampling at the data locations and time-averaging of the model output); the superscript \( T \) denotes matrix transpose; and \( C_d = \sigma_d^2 I \) is the data error covariance (where \( I \) is the unity matrix). The following form is assumed for the bathymetric error covariance:

\[
C_h(x_1, y_1, x, y) = \sigma_h^2 \alpha(x_1, y_1) \alpha(x, y) \exp\left[-\frac{(y - y_1)^2}{2 l_y^2}\right] \exp\left[-\frac{1 - \cos[q(x - x_1)]}{(q l_x)^2}\right],
\]
where

\[
\alpha(x, y) = \begin{cases} 
\sin^2 \left( \frac{\pi}{2} \frac{h(x, y)}{h_{\text{min}}} \right), & \text{if } h < h_{\text{min}}, \\
1, & \text{if } h \geq h_{\text{min}},
\end{cases}
\]  

(7)

\[ q = 2\pi / L_X, \]  and \( L_X \) is the periodic channel length. For given \( x_1 \) and \( y_1 \), the product of the exponential functions in (6) is a bell-shaped function in \( x \) and \( y \). The functional dependence in \( x \) conveniently accounts for the domain periodicity (Ménard, 2005). Note that this particular functional form is not influenced by knowledge of the true bathymetry in our case (the cosine function in (6) emerges if ends of a periodic domain are connected and the domain presented as a circle; then the correlation is defined as a Gaussian function of the distance between points on the circle). The scaling function \( \alpha (0 \leq \alpha \leq 1) \), equation (7), is to reduce error standard deviation (and hence the magnitude of the correction) in very shallow water, to possibly avoid drying. The following parameters were used in our experiments: \( \sigma_h = 0.05 \) m, \( h_{\text{min}} = 1.5 \) m, \( l_X = 50 \) m, and \( l_Y = 20 \) m.

Minimization of (7) is done using the indirect representee method (Chua and Bennett, 2001, Bennett, 2002, Kurapov et al., 2007, 2011). Without discussing every detail, we mention here that the original nonlinear optimization problem is linearized (with respect to the prior solution) and the correction is obtained as an optimal linear combination of representer functions:

\[
h^{\text{INVERSE}} = h^{\text{PRIOR}} + \delta h = h^{\text{PRIOR}} + \sum_{k=1}^{K} b_k \delta h_k, \]  

(8)
where $\delta h_k$ is the bathymetry component of the representer corresponding to the $k$-th observation, $b_k$ are representer coefficients, and $K$ is the total number of data. Fields $\delta h_k$ show zones of influence of each velocity observation on the bathymetry. To compute each $\delta h_k$, the AD model can be forced with the impulse at the corresponding observational location and time. Since the time-averaged current data is assimilated, a fraction of the impulse is added to the velocity component of the adjoint solution multiple times spread over the period of (1,2) h. The AD model is run backward in time over the interval $(0, 2)$ h and returns adjoint sensitivity in the initial conditions $\lambda_k^u(0)$ and bathymetry $\lambda_k^h$, corresponding to the $k$-th observation. Then

$$\delta h_k = C_h \lambda_k^h.$$ If the coupled TL model is started with $\delta h_k$ and zero initial condition perturbation (corresponding to the exact initial conditions in our case), the output will be a representer $r_k$, which would include fields corresponding to each TL model output component (such as wave vector components, wave energy, and the time-varying current fields). Sampling the full set of representers at observation locations and time, we obtain a symmetric and positive-definite representer matrix: $R = L[r_1|\cdots|r_K]$, of size $K \times K$. Then, the optimal set of representer coefficients $b = \{b_k\}$ can be obtained solving the linear algebraic system as follows:

$$\begin{align*}
(R + C_d)b &= d - Lu^{\text{PRIOR}}. \\
&= (R + C_d)b = d - Lu^{\text{PRIOR}}.
\end{align*}$$

(9)

In the indirect representer method (see Bennett, 2002), which is designed for large data sets, it is not necessary to compute each representer. The initial guess about $b$ is made. The AD system is forced with a linear combination of impulse forcings at data locations and times, each scaled by $b_k$. The bathymetry component of the AD model output is convolved with the
covariance \( C_h \), to obtain an estimate of \( \delta h \). The TL model is then run given \( \delta h \) and the zero initial condition perturbation. The TL model output is sampled at the observation locations, using operator \( L \), and the resulting vector yields the product \( Rb \), which is used in the conjugate gradient method to solve (9) iteratively. This product, the data error covariance, and the prior model-data misfit [the rhs of (9)] are utilized at every iteration to obtain a better estimate of \( b \). This AD-TL cycle is repeated until a specified convergence criterion is satisfied:

\[
\frac{\| (R + C_d) b - (d - Lu^{\text{PRIOR}}) \|^2}{\| d - Lu^{\text{PRIOR}} \|^2} \leq 10^{-3}.
\]

Three tests have been considered. Plots of inverted bathymetries in each test are shown in Figure 9. In the first (Figure 9a), the flow was weakly nonlinear (\( r = 0.004 \) m s\(^{-1}\)) and both velocity components were assimilated at each data point. In the second (Figure 9b), the flow was strongly nonlinear (\( r = 0.002 \) m s\(^{-1}\)) and both velocity components were assimilated. In the third (Figure 9c), the flow was weakly nonlinear (\( r = 0.004 \) m s\(^{-1}\)) and only the alongshore velocity component \( u \) was assimilated at each data point, simulating the observational sets that can be obtained from optical imagery (Chickadel et al., 2003). In every case the bathymetry is improved qualitatively and also quantitatively, in terms of the area-averaged root mean square (RMS) error (Table 1). In every case, the bar is moved toward the shore and is shaped to correspond more closely to the true bathymetry. The displacement in the bar location is best seen in alongshore-averaged bathymetry profiles (Figure 10). In the case of the weakly nonlinear flow, the result assimilating only the \( u \) velocity component is as good as that assimilating both velocity components (e.g., compare RMS statistics and Figure 10a and c). In the case of the
strongly nonlinear flow (Figure 10b), although the bathymetry is generally improved, the slope offshore of the bar is not quite correct.

4. Representer and array mode analysis; experiments with different observational arrays

In this section, analyses of the $h$-component of the adjoint sensitivity field $\lambda_k^{(h)}$, the $h$-component of the representer $\delta h_k$, and the correction $\delta h$ (which is the optimal linear combination of $\delta h_k$) are discussed with the goal of understanding relative contributions of the wave and circulation models to the bathymetry correction. The results from the weakly nonlinear case assimilating both components of velocity are used for illustration.

The plots of $\lambda_k^{(h,W)}$, $\lambda_k^{(h,C)}$, and $\lambda_k^{(h)} = \lambda_k^{(h,W)} + \lambda_k^{(h,C)}$ corresponding to a single observation of the alongshore velocity at a location over the bar are shown in Figure 11. Velocity-bathymetry sensitivity $\lambda_k^{(h)}$ may be interpreted as the model error covariance of the bathymetry $h(x, y)$ and the time-averaged velocity component at the observed location under the assumption that bathymetric errors are spatially uncorrelated ($C_h = I$; see Bennett, 2002). The wave and circulation constituents $\lambda_k^{(h,W)}$ and $\lambda_k^{(h,C)}$ have different spatial structures. The output of the adjoint circulation model $\lambda_k^{(h,C)}$ is nearly singular, showing large sensitivity of the velocity to local changes in bathymetry. The singular behavior of the adjoint sensitivity field is a common occurrence in advective transport systems (Chua and Bennett, 2001; Bennett, 2002). If the color scale was adjusted, we could also see some influence of the bathymetry along the bar (as the effect of advection by the background current). The alongshore advection in AD_C also impacts the forcing of the adjoint wave model $\lambda_k^{(f)}$, which results in enhanced sensitivity along the bar location ($80<y<100$ m) in $\lambda_k^{(h,W)}$ (the output of the adjoint wave model). Also, this field shows
sensitivity of the velocity to bathymetry changes farther offshore, in a general direction of the background wave propagation. This pattern could be expected, as errors (corrections) in the bathymetry offshore of the observation location would affect propagation and dissipation of the wave, hence the forcing of the circulation model and the resulting velocity at the observed location.

Applying the covariance $C_h$ to each of the two terms of $\lambda_k^{(h)}$ separately, and to the sum, the corresponding fields of $\delta h_k^{W}$, $\delta h_k^{C}$, and $\delta h_k$ are obtained (Figure 12). The singularity at the observation location is smoothed as a result of the covariance implementation. For the selected observation, the structure of the total $\delta h_k$ is dominated by its wave constituent, although the circulation term provides a measurable contribution in the area over the bar (where its magnitude is about 30% of that of the wave model term). Relative contributions $\delta h_k^{W}$ and $\delta h_k^{C}$ to the bathymetric correction may depend on the observation location. The co-variability pattern along the wave characteristics in deeper water is preserved in both $\delta h_k^{W}$ and $\delta h_k^{C}$. This suggests a potential for correction (or at least an impact) in the deep water area not covered by the data. In our test case, the true and prior bathymetries are very similar far offshore, where a correction would not be needed. Remarkably, despite the fact that the bathymetry error standard deviation $\alpha(x,y)$ (7) is not reduced in deep water, the offshore impacts from each representer nearly cancel in the optimal combination (8) such that the net bathymetric correction is minimal far offshore (although small undulations can still be seen in the inverse bathymetry at $y>250$ m, see Figure 9).

To understand relative contributions of the circulation and wave adjoint model components to the bathymetry correction given the whole observational array, we will next analyze the optimal linear combinations
as well as the total bathymetric correction \( \delta h = \delta h_W + \delta h_C \) [the second term on the rhs of (8)] (Figure 13). The term \( \delta h_W \) (Figure 13a) is positive (negative) in the area offshore (onshore) of the prior bar location. This means that the adjoint wave model contributes to the bathymetric correction mostly by displacing the bar inshore. The term \( \delta h_C \) (Figure 13b), provided directly by the adjoint circulation model, is the largest in the area of the true bar location, additionally deepening the bathymetry at the left and right sides of the bar. Thus, in our case, the two adjoint model components have distinct roles. While AD_W is responsible for displacing the bar, AD_C shapes the bar, varying bathymetry in the alongshore direction. This is consistent with our understanding that the wave properties are more sensitive to the location of the bar than low-amplitude alongshore variability in the bar shape. In turn, time-averaged currents are sensitive to the alongshore variability in the bar shape.

It should be noted that the output of the adjoint wave model \( \delta h_W \) is not independent from the adjoint circulation model. The AD_C component of the coupled assimilation system provides sensitivity of the velocity to the wave model forcing \( \lambda(f) \), see Figure 2c), which is utilized to force AD_W. In broad terms, this sensitivity contains information about where the largest impact of the waves on the circulation can be.

A similar result, showing different spatial patterns in \( \delta h_W \) and \( \delta h_C \), is obtained in the strongly nonlinear case (Figure 14). The deepening of the slope offshore of the true bar location is not as pronounced as in the weakly nonlinear case, in part because the true-prior velocity difference in the area of the bar is smaller in the strongly nonlinear case (see Figure 8).
The representer method also allows us to perform array mode analysis (Egbert 1997, Bennett, 2002, Kurapov et al., 2009) to identify spatial structures that can be corrected most stably, given errors in the data. This is illustrated here using the weakly nonlinear case, assimilating both components of velocity. The analysis is based on the eigenvalue decomposition of \( R + C_d \). In our case, there are only 1250 observations of the time-averaged \( u \) and \( v \), such that computation of all the representers is possible resulting in the full representer matrix. Since \( C_d = \sigma_d^2 I \) in our case, we can compute the singular vector decomposition of \( R = USU^T \) and re-write the equation for the optimal representer coefficients (8) as follows:

\[
b = U(S + \sigma_d^2 I)^{-1}U^T(d - Lu^{\text{PRIOR}}). \tag{11}
\]

Elements of the vector \( U^T d \) are array modes obtained as linear combinations of the original data set, and \( U^T Lu^{\text{PRIOR}} \) is their model counterpart. Given the prior assumptions about the model input errors (in particular \( C_h \)), the diagonal matrix \( S \) is the prior covariance of errors in elements of the vector \( U^T Lu^{\text{PRIOR}} \). The diagonal elements \( s_k \) of this matrix yield the expected prior model error variance in each element of \( U^T Lu^{\text{PRIOR}} \) (Figure 15). The linear combinations for which \( s_k > \sigma_d^2 \) can be corrected stably (with respect to errors in the data). We find that in our case only 18 modes satisfy this requirement. The bathymetry correction structure corresponding to each of these modes is obtained as \( \sum_{k=1}^{K} U_{ki} \delta h_k \), where \( i \) is the mode number (Figure 16). In the first 4 of these most stably observed structures, correction in the area of the bar is strongly coupled with that in the area not covered by the data. These 4 modes would be stably constrained even in the case of very imprecise data (e.g., with \( \sigma_d = 0.2 \) m, see Figure 15). Mode 5-8 bathymetry correction structures are locally intensified in the area of the bar and onshore of it. Higher modes
are responsible for correction on smaller spatial scales, which exhibit some weak coupling with
the distant correction farther offshore.

The fact that only a handful of the model modes can be effectively corrected encouraged us
to explore how effective assimilation may be with smaller data sets. These additional
observational arrays and posterior bathymetries resulting from their assimilation are shown in
Figure 17c-f. For reference, the true and prior bathymetries are also shown in Figure 17a-b.
These results correspond to the weakly nonlinear case. In all the cases, the \((u,v)\) data are sampled
at points on regular grids, which differ in the offshore extent \((Y)\) and the distance between the
data points in each direction \((DX)\). These assimilation cases are labeled as \(Y-DX\); for instance
case 256-10 (Figure 17c) is one of the initial cases, presented earlier. The RMS errors of the
inverse bathymetry estimates for each of the cases presented in Figure 17 are shown in Figure
18. It turns out that using data only onshore and over the bar and discarding the points at
\(y > 128\) m (case 128-10, Figure 17d) results in a bathymetry estimate of almost the same
(slightly better in terms of RMSE) quality as the initial case 256-10. Sampling the velocities
every 20 m (instead of originally 10 m) in the same smaller area (case 128-20, Figure 17e),
degraded the result of the inversion, although this estimate is still a substantial improvement
compared to the prior. Case 64-10, using a dense set of data only onshore of the bar trough
(Figure 17f), results in an estimate of \(h\) that is improved over the prior, albeit only marginally. In
that case, the alongshore variability in the bar shape is improved, but it is not moved toward the
shore.
5. Summary

Opportunities for bathymetry correction provided by the variational data assimilation method have been explored in the context of a one-way coupled nearshore wave-circulation model. Tests using synthetic data show that assimilation of velocity observations can provide a useful correction to bathymetry. The variational formalism allows separating contributions to bathymetry correction coming directly from the adjoint circulation model and adjoint wave model. In our case, the wave component was responsible for the bar displacement toward the coast, and the circulation model for alongshore variability in the bar shape.

Our idealized study discussed advantages of using the variational method, which include a very clear formulation of the optimization problem, a framework for providing bathymetry correction at locations where observations are not available, the possibility to determine contributions from different model components in the coupled system, and the analysis of best constrained spatial structures (array modes). We found that individual representers show local correction as well as correction to bathymetry in areas offshore along the propagation direction of the waves. In our case, the prior bathymetry estimate was very accurate offshore, and remarkably, the optimal linear combination of the individual representers results in cancellations such that the offshore areas of the prior bathymetry are not altered significantly.

Analysis of the array modes suggests that the bathymetric correction can be constrained with relatively few linear combinations of observations. Tests with truncated observational arrays confirm that qualitative and quantitative improvement of the bathymetry can be obtained with observational arrays that mimic those used for remotely sensed (e.g. video-based) surf zone current observations.
Whether this model is usable in realistic cases will depend on the situation. To make it more useful for a range of practical nearshore problems, the system may be advanced by incorporating the following elements: a wave model describing an evolution of a full spectrum of waves (such as SWAN, Veeramony et al., 2010), a three-dimensional circulation model and its TL and AD components (Moore et al., 2011, Kurapov et al., 2011, Yu et al., 2012), and two-way coupling (Uchiyama et al., 2009), and two-way coupling between the wave and circulation models.

**Appendix. The wave model**

Here, for reference, the wave model by Slinn et al. (2000) is briefly described, taking into account modification (4). Given the root-mean-square wave height $H_{rms}$ and direction of the incoming waves $\theta$ in deep water, these parameters at the offshore boundary of our domain (6.1 m depth) are determined using Snell’s law and conservation of wave energy. At any depth, the wave number magnitude $\kappa = (k^2 + l^2)^{1/2}$ is found by inverting the dispersion relation:

$$\omega^2 = g \kappa \tanh[\kappa h],$$  \hspace{1cm} (A.1)

where, recall, $\omega = 2\pi/T_p$ is the peak wave frequency. Given the conditions at the offshore boundary, (4) is integrated numerically in the direction toward coast using the 4th order Runge-Kutta method:

$$\frac{\partial k}{\partial y} = \frac{\partial l(k, \omega, h)}{\partial x},$$  \hspace{1cm} (A.2)
where functional dependency of \( l \) on \( k, \omega \) and \( h \) is defined by (A.1). The magnitude of the phase velocity is

\[
C = \frac{\omega}{\kappa}, \tag{A.3}
\]

The magnitude of the group velocity is

\[
C_g = \frac{\omega}{2k} \left( 1 + \frac{2kh}{\sinh 2h} \right), \tag{A.4}
\]

and its components are:

\[
C_g^{(x)} = C_g \frac{k}{\kappa}, \tag{A.5}
\]

\[
C_g^{(y)} = C_g \frac{l}{\kappa}.
\]

To obtain the wave energy \( E(x,y) \), the corresponding equation is integrated (from the offshore boundary, using the Runge-Kutta method):

\[
\frac{\partial (EC_g^{(x)})}{\partial x} + \frac{\partial (EC_g^{(y)})}{\partial y} = -\varepsilon_b, \tag{A.6}
\]

where the dissipation function is

\[
\varepsilon_b = \frac{3\sqrt{\pi} \rho g B^3}{16T_p} \frac{H_{rms}^3}{\gamma^2 h^3} \left\{ 1 - \left[ 1 + \left( \frac{H_{rms}}{\gamma h} \right)^2 \right]^{-5/2} \right\}, \tag{A.7}
\]

\[
E = \frac{\rho g H_{rms}^2}{8}. \tag{A.8}
\]
As in Slinn et al. (2000), \( B = 1.2 \) and \( \gamma = 0.43 \). The components of the radiation stress tensor are:

\[
S_{xx} = \frac{E}{2} \left( \frac{2C_g}{C} \cos^2 \theta + \frac{2C_g}{C} - 1 \right)
\]

\[
S_{xy} = \frac{E}{2} \frac{C_g}{C} \cos \theta \sin \theta
\]

\[
S_{yy} = \frac{E}{2} \left( \frac{2C_g}{C} \sin^2 \theta + \frac{2C_g}{C} - 1 \right)
\]

where

\[
\cos \theta = \frac{k}{\kappa}, \quad \sin \theta = \frac{l}{\kappa}.
\]

Finally, the forcing of the momentum equations (2)-(3) is computed as:

\[
f_x(x, y) = -\frac{1}{\rho h} \left[ \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} \right],
\]

\[
f_y(x, y) = -\frac{1}{\rho h} \left[ \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} \right].
\]

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References.


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Figure captions.

Figure 1. Model bathymetry in twin experiments: (a) true, (b) prior. Black contours are every 0.5 m.

Figure 2. The schematics of the coupled model components: (a) nonlinear, (b) tangent linear (information flow is similar to the nonlinear model), and (c) adjoint (information flow is reversed compared to the nonlinear and tangent linear models).

Figure 3. Snapshots of velocity (vectors) and vorticity (contours) at three time instances (left to right: 1.5, 1.75 and 2 h since the start from rest). Upper plots (rows 1 and 2): $r = 0.004$ m/s, lower plots (rows 3 and 4): $r = 0.002$ m/s. Half-tone contours in the background of the velocity plots are $h = 2$ m (showing the location of the bar between 70 and 100 m offshore); vorticity contours are every 0.01 s$^{-1}$ (top) / 0.02 s$^{-1}$ (bottom), with positive values shaded.

Figure 4. The relative difference (%) in the cross-shore component of wave celerity computed over the true and prior $h$.

Figure 5. The alongshore current averaged in space in the alongshore direction and in time (over the interval of (1, 2) h); the solid (dashed) line corresponds to the current over the true (prior) bathymetry: (a) $r = 0.004$ m/s, (b) $r = 0.002$ m/s.

Figure 6. Observation locations in the synthetic data assimilation experiment. The half-tone contours are the 2-m isobaths (showing the location of the bar in the true bathymetry case).

Figure 7. Time-averaged ($t=1-2$ h) currents shown at every 2nd observational location: (top) the weakly nonlinear case, (bottom) the strongly nonlinear case, (left) the true solution, (right) the prior solution. Half-tone lines are bathymetry (every 0.5 m).
Figure 8. The magnitude of the difference of the true and prior time-averaged velocities

\[(\bar{u}^\text{TRUE} - \bar{u}^\text{PRIOR})^2 + (\bar{v}^\text{TRUE} - \bar{v}^\text{PRIOR})^2\], where \(\bar{u}\) and \(\bar{v}\) correspond to the velocity components averaged over interval \(t = (1,2)\) h: (a) the weakly nonlinear case, (b) the strongly nonlinear case.

Figure 9. Inverse bathymetry: (a) \(r = 0.004\) m/s, assimilating \(u\) and \(v\) components, (a) \(r = 0.002\) m/s, assimilating \(u\) and \(v\) components, (c) \(r = 0.004\) m/s, assimilating the alongshore (\(u\)) component only. The corresponding RMS errors with respect to the true bathymetry are listed in Table 1. For alongshore average bathymetry profiles see Figure 10.

Figure 10. Alongshore-average bathymetry profiles, true (solid black), prior (dashed), and inverse (half-tone): (a) \(r = 0.004\) m/s, assimilating \(u\) and \(v\) components, (b) \(r = 0.002\) m/s, assimilating \(u\) and \(v\) components, (c) \(r = 0.004\) m/s, assimilating the alongshore (\(u\)) component only.

Figure 11. The adjoint bathymetry field corresponding to a single observation, resulting from the adjoint representer computation; units are \((\text{m s}^{-1})\) m\(^{-1}\); the observation location is at the point where the solution is nearly singular: (a) \(\lambda_k^{(h,W)}\), (b) \(\lambda_k^{(h,C)}\), (c) \(\lambda_k^{(h)} = \lambda_k^{(h,W)} + \lambda_k^{(h,C)}\).

Figure 12. The representer bathymetry components, obtained by applying the prior error covariance \(C_h\) to the adjoint fields shown in Figure 11; contours are every \(5 \times 10^{-5}\) m\(^2\) s\(^{-1}\); the observation location is shown as the circle: (a) \(\delta h_k^W\), (b) \(\delta h_k^C\), (c) \(\delta h_k = \delta h_k^W + \delta h_k^C\).

Figure 13. Contributions of the wave and circulation models to the bathymetric correction and the total correction (meters): (a) \(\delta h_W\), (b) \(\delta h_C\), (c) \(\delta h = \delta h_W + \delta h_C\). The black lines are the 2-m isobaths in the prior bathymetry (the two lines offshore show the location of the bar). The case is with \(r = 0.004\) m/s, assimilating \(u\) and \(v\) components.
Figure 14. Similar to Figure 13, for the strongly nonlinear case ($r = 0.002$ m/s). Note a different (finer) scale of the color bar.

Figure 15. The representer matrix spectrum (variances of the first 50 array modes). The dashed line corresponds to $\sigma_d^2$.

Figure 16. Bathymetry correction functions corresponding to the first 12 array modes.

Figure 17. Bathymetries in additional assimilation tests exploring impacts of different data arrays (dots): (a) true, (b) prior, (c) case 256-10 (i.e., data extend to $y = 256$ m, and the distance between data points is 10 m in both alongshore and cross-shore directions), (d) case 128-10, (e) case 128-20, (f) case 64-10. Bathymetric contours are shown every 0.5 m. RMSE corresponding to these cases is shown in Figure 18.

Figure 18. Bathymetry RMSE with respect to the true $h$, for the cases shown in Figure 17.
Table 1. The bathymetric RMS error (cm), with respect to the true bathymetry, of the prior and the three inverse estimates obtained in the assimilation tests using the observational array as in Figure 6.

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<td>Prior $h$</td>
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<td>Inverse $h$, weakly nonlinear flow, $u$ observations</td>
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- $\delta h^W_k$ 
- $\delta h^C_k$ 
- $\delta h_k$
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