Representer-based variational data assimilation in a nonlinear model of nearshore circulation

Alexander L. Kurapov,1 Gary D. Egbert,1 J. S. Allen,1 and Robert N. Miller1

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[1] A representer-based variational data assimilation (DA) method is implemented with a shallow-water model of circulation in the nearshore surf zone and tested with synthetic data. The behavior of the DA system is evaluated over a 1-hour time interval that is large compared to timescales characteristic of instability growth and eddy interactions. True reference solutions, from which the synthetic data are sampled, correspond to fully developed unsteady nonlinear flows driven by a steady spatially varying forcing representing the effect of breaking waves. Forcing and initial conditions are adjusted to fit the data. The convergence of the nonlinear optimization algorithm and the accuracy of the forcing and state estimates depend on the choice of the forcing error covariance C. In a weakly nonlinear (equilibrated waves) regime, using C that allows only a steady forcing correction yields a convergent and accurate solution. In a more strongly nonlinear regime, the DA system cannot find sufficient degrees of freedom in the steady forcing to control eddy variability. Implementing a bell-shaped temporal correlation function in C with the 1-min decorrelation scale yields a convergent linearized inverse solution that describes correctly the spatiotemporal variability in the eddy field. The corresponding estimate of forcing, however, is not satisfactory. Accurate estimates of both the flow and the forcing can be achieved by implementing a composite C with a temporal correlation separated into an O(1) steady and small amplitude time-variable parts.


1. Introduction

[2] Variational data assimilation (DA) methods have been implemented in oceanography for synthesis of observations and models with goals to quantify and to improve understanding of ocean dynamical processes and forcing mechanisms [e.g., Egbert et al., 1994; Bennett et al., 1998, 2000, 2006; Egbert and Ray, 2000; Bennett, 2002; Yaremchuk and Maximenko, 2002; Kurapov et al., 2003; Foreman et al., 2004; Stammer et al., 2004; Stammer, 2005; Wunsch and Heimbach, 2007]. There are also advantages in utilizing variational DA for operational applications, to provide improved initial conditions for forecast models [e.g., Rabier and Courtier, 1992; Ehrendorfer, 1992; Rosmond and Xu, 2006; Di Lorenzo et al., 2007]. Variational methods find an estimate of a system state that fits a dynamical model and available observations in a least squares sense. The optimal solution minimizes a penalty functional that is a sum of squared norms of residuals (deviations from the prior estimates) in model inputs and data, integrated over the computational domain and specified time interval. The penalized model inputs may include initial and boundary conditions, external forcing, model parameters, and interior dynamical errors. Because of nonlinearities in computational ocean models, variational oceanographic problems are generally solved iteratively, utilizing companion tangent linear and adjoint numerical codes.

[3] The variational representer-based method [Chua and Bennett, 2001; Bennett, 2002] finds a minimizer of the penalty functional by solving a series of linearized optimization problems. On each iteration, a correction to the prior solution is sought in the subspace spanned by the representer functions corresponding to the observations. The representer method offers greater flexibility in the choice of norms of the model penalty terms than the so called adjoint method or 4DVAR [e.g., Wunsch and Heimbach, 2007], providing means for making explicit and nontrivial statistical hypotheses about prior errors in the model inputs. For instance, by choosing the norms, one can specify nontrivial spatial and temporal correlations among input errors, as well as additional dynamical constraints on corrections at the initial time [Rosmond and Xu, 2006] or at the open boundaries [Kurapov et al., 2003]. These choices may affect both the computational efficiency of the optimization algorithm and the inverse solution accuracy, as shown here.

[4] The objective of the present study is to assess the utility of the representer method implemented for a nonlinear model of forced dissipative oceanic flows, in which spatiotemporal variability is dominated by instabilities and interior eddy interactions. Such behavior can be found, for
example, in two-dimensional (shallow water) models of circulation in the nearshore surf zone [Allen et al., 1996; Slinn et al., 2000] and in high-resolution three-dimensional models of stratified shelf flows [Durski and Allen, 2005]. Previous studies on variational DA in the presence of instabilities have mostly been focused on applications in which the assimilation interval was comparable to the timescale of linear instability growth and the initial conditions were a major source of error [e.g., Rabier and Courtier, 1992; Xu and Daley, 2002]. In our study, the representer method is applied over time intervals that exceed substantially both the characteristic timescale of linear instability growth or the timescales associated with eddy motions (which can be evaluated using the frequency spectrum, see below). Assimilation in such long time windows may be necessary to obtain an accurate estimate of forcing that varies on a slower timescale. If temporal flow variability is a result of intrinsic nonlinear interactions rather than a direct effect of time-variable forcing, it is not obvious that the variational method, trying to fit the data by adjusting the forcing, would result in a convergent or accurate solution. In our study, we explore the impact of the forcing error covariance, which must be specified prior to assimilation, on solution convergence and on the accuracy of the inverse estimates of both the flow and forcing.

[5] To explore these issues, the representer method is implemented with an idealized shallow-water model of nearshore circulation over variable beach bathymetry and tested on synthetic data generated by the nonlinear model. The dynamical content is close in many aspects to that of Slinn et al. [2000]. The alongshore current is forced by gradients in the radiation stresses resulting from breaking waves. Over the time periods considered, the forcing is assumed to be steady. If the linear friction coefficient is sufficiently small, the current is subject to shear instabilities that evolve to a weakly nonlinear equilibrated shear wave regime or, as friction is reduced further, to a more strongly nonlinear irregular eddy regime. We consider DA experiments in both unsteady regimes.

[6] Aside from the pioneering study by Feddersen et al. [2004] who assimilated time-averaged observations in a linear one-dimensional (cross-shore coordinate) model, there has been little prior work on DA in the nearshore ocean. However, advanced DA methods might be used to combine existing extensive field observations with models providing better understanding of some of the complex flow processes in the nearshore region. Although our focus here is on fundamental aspects of DA in a nonlinear oceanic system, this study certainly represents a next step toward DA in the nearshore surf zone by utilizing a model that would be more directly comparable to the time-variable observed fields than the one-dimensional model.

2. A Dynamical Model

[7] A Cartesian coordinate system \((x, y)\) is utilized, with the \(x\) axis aligned alongshore, the \(y\) axis positive in the offshore direction, and the origin at the shore. The model domain is rectangular, \(256 \times 200\) m, representing a beach area with a straight coastline and along- and cross-shore variable bathymetry including a sand bar parallel to the coast (Figure 1). The depth is approximately 0.06 m at interior grid points closest to the coast and 4.1 m near the offshore boundary. No normal flow boundary conditions are applied at the coast and offshore boundaries; periodic boundary conditions are utilized in the alongshore direction. The dynamics are described by the nonlinear shallow-water equations

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (Du)}{\partial x} + \frac{\partial (Dv)}{\partial y} = 0,
\]

\[
\frac{\partial (Du)}{\partial t} + \frac{\partial (Du u)}{\partial x} + \frac{\partial (Dv u)}{\partial y} = -gD \frac{\partial \zeta}{\partial x} + f_x - ru - a \nabla^2 (H \nabla^2 u),
\]

\[
\frac{\partial (Dv)}{\partial t} + \frac{\partial (Du v)}{\partial x} + \frac{\partial (Dv v)}{\partial y} = -gD \frac{\partial \zeta}{\partial y} + f_y - rv - a \nabla^2 (H \nabla^2 v),
\]

where \(\zeta(x, t)\) is the free surface elevation, \(u(x, t)\) and \(v(x, t)\) are, respectively, the alongshore and cross-shore components of the depth-averaged velocity, \(x = (x, y)\), \(H(x)\) is the bathymetric depth, \(D = H + \zeta\) is the total water depth, \(g\) is the gravitational acceleration, \(f_x(x, y)\) and \(f_y(x, y)\) are forcing components, and \(r\) is the linear friction coefficient. Weak biharmonic horizontal friction terms, with \(a = 1.25\) m\(^2\) s\(^{-1}\) as in work by Slinn et al. [2000], are included in (2) and (3) to provide numerical dissipation at small length scales.

[8] Details of the space and time discretization are adopted from the Regional Ocean Modeling System (ROMS) [Haidvogel et al., 2000]. The equations are discretized on a staggered rectangular C-grid, with a second-order centered difference scheme utilized for the advection term. The equations are integrated in time using a predictor-corrector (leapfrog-Adams-Bashforth) scheme. For computations discussed here, the grid resolution is 2 m in either direction and the time step is 0.15 s. At the coast and offshore boundaries, in addition to the condition \(v = 0\), we set \(\frac{\partial^2 u}{\partial y^2} = 0\), \(\partial u/\partial y = 0\), and \(\partial^2 u/\partial y^2 = 0\) when discretizing the biharmonic terms [Allen et al., 1996].
To obtain the true solution for our DA experiments, the nonlinear model forcing \( f \{X, Y\}(x) \) is steady. This approximates the effect of obliquely incident breaking waves and is computed using a wave model as described by Slinn et al. [2000, Appendix C], utilizing the Thornton and Guza [1983] formulation for wave energy dissipation. The forcing is computed in a larger domain extending 512 m in the offshore direction, to deeper water. The surface waves are assumed to arrive in the nearshore zone obliquely from the right. The angle between the direction of wave phase propagation in deep water far offshore and the \( x \) axis is \( 135^\circ \). At \( y = 512 \) m, the wave rms height is 0.7 m, and the peak period of the narrow band wave energy spectrum is 8 s. On average, the alongshore component \( f_X \) (Figures 2a and 2b) forces an alongshore current in the negative direction (from right to left Figures 2a and 2b). The cross-shore component \( f_Y \) (Figures 2c and 2d) supports the elevated level (set-up) of the water near the coast. The magnitude of both \( f_X \) and \( f_Y \) is larger over the offshore side of the bar and over the slope near the coast, although it is sharply reduced to 0 very close to the coast.

The nonlinear model is spun up from rest. After an initial adjustment the flow regime depends on the magnitude of the friction coefficient \( r \). If \( r \) is large enough, the flow becomes steady and the dominant dynamical balance is linear, between the forcing and the bottom friction. For lower values of \( r \) the alongshore current may develop shear instabilities. Two cases in which the well-developed flow is unsteady are considered here, with \( r = 0.004 \) m s\(^{-1}\) and \( r = 0.002 \) m s\(^{-1}\). After approximately two model hours, in the first case the area-averaged, depth-integrated kinetic energy asymptotes to a constant value, while in the second case it approaches a state with irregular 10-min timescale fluctuations about a larger constant value (Figure 3). In the case \( r = 0.004 \) m s\(^{-1}\), the model solution evolves into a weakly nonlinear equilibrated shear wave regime. The instantaneous velocity and vorticity \( (\xi = \partial v/\partial x - \partial u/\partial y) \) fields at \( t = 5 \) hours corresponding to this regime are shown in Figures 4b and 4c. After the solution equilibrates, this flow structure translates from right to left disturbed only by very minor variations in the intensity of the eddies. The dominant shear wave period is near 6 min. The alongshore current averaged in time and in the alongshore direction (Figure 4a) has peaks near the coast (0.8 m s\(^{-1}\)) and over the bar (0.4 m s\(^{-1}\)). In the more strongly nonlinear case with \( r = 0.002 \) m s\(^{-1}\) (Figures 4d, 4e, and 4f), the flow can be described as irregular, with more intense eddies and a more energetic mean alongshore current; the peak values are now 1.2 m s\(^{-1}\) near the coast and 0.5 m s\(^{-1}\) near the sand bar.

To provide a low-dimensional representation of the two solutions that emphasizes the difference in their nonlinear behavior, the kinetic energy of deviations from the time mean is analyzed in the 4-hour interval beginning at \( t = \)
5 hours. In Figure 5, the depth-integrated and area-averaged kinetic energy of the velocity deviations, \(K\), is plotted against \(dK/dt\). The trajectory corresponding to the case \(r = 0.004\) m s\(^{-1}\) is nearly a point compared to the irregular trajectory for the case \(r = 0.002\) m s\(^{-1}\). In the first case, the domain-averaged energy of the time-variable part of the flow is preserved, which is characteristic of the equilibrated shear waves, while in the second, more strongly nonlinear case there are substantial irregular fluctuations in \(K\) and \(dK/dt\).

The frequency-wave number \((\omega - \kappa)\) power spectra of the vorticity \(\xi(x, t)\) corresponding to the 1-hour time interval between \(t = 5\) and 6 hours are shown in Figure 6 for alongshore profiles \(y = 8\) m (nearshore slope), 38 m (trough between the shore and the bar), 65 m (the bar), and 98 m (offshore from the bar). In the equilibrated wave case, the spectra are discrete (Figures 6a–6d), with most power concentrated near \((\omega, \kappa) = (0.003, 0.01)\) (cycles s\(^{-1}\), m\(^{-1}\)), corresponding to motions with a nearly 6-min period and 100-m wavelength. In the more irregular case the spectra are continuous (Figures 6e–6h), with substantial power in the motions near \(\omega = 1/(60s) = 0.017\) s\(^{-1}\) on the inshore side of the bar (Figures 6e and 6f). The dominant phase speed \(\omega/\kappa\) is 0.75 m s\(^{-1}\) on the inshore side and 0.5 m s\(^{-1}\) on the offshore side of the bar.

3. Data Assimilation (DA) Problem
3.1. Penalty Functional

In the following, it is convenient to denote the state as a vector field, \(u(x, t) = [\zeta u v]\) (the prime denotes matrix transpose); the two forcing components are also combined in a vector field \(f(x, t) = [f_x f_y]\). In our experiments, the true model starts from initial conditions corresponding to the fully developed flow, sampled from the nonlinear solutions from the time-mean current obtained from the nonlinear solutions at \(t = 5\)–9 hours. To help the viewer see the trajectories, line intensity changes from light (earlier times) to dark (later times).
described above at \( t = 5 \) h (see Figure 4). To explore the nonlinear aspect of the optimization algorithm, we deliberately choose prior inputs that differ substantially from the truth, namely, \( u_{\text{prior}}(x,0) = u_{0} \) and \( f_{\text{prior}} = 0 \) (the prior state is thus 0). The time series observations of \( z, u, \) and \( v \) are sampled from the true solution at 32 locations shown in Figure 1 once every minute from \( t = 1 \) to \( 59 \) min; the total number of observations is \( K = 5,664 \). Random noise is added to the observations with standard deviation \( \sigma_d = 0.02 \) m for \( z \) and \( 0.02 \) m s\(^{-1} \) for \( u \) and \( v \).

In symbolic form, the nonlinear (NL) model can be written as

\[
\frac{\partial u}{\partial t} = N(u) + f_{\text{prior}} + e \tag{4}
\]

\[
u(x,0) = u_{0}^{\text{prior}} + e_{0} \tag{5}
\]

where \( u \) is the true state, \( N \) is the NL model operator, \( e(x,t) \) is the forcing error, and \( e_{0}(x) \) is the initial condition error. The observations are written in the general form as

\[
L(u) = \int_{S} dx \int_{0}^{T} dt \begin{pmatrix}
g_{1}(x_1,t_1;x,t) \\
g_{2} \\
\vdots \\
g_{K}
\end{pmatrix} u(x_1,t_1) = d + e_{d} \tag{6}
\]

where \([0, T]\) is the assimilation time interval, \( S \) is a domain area, \( dx = dx \, dy \), \( d = \{d_k\} \) is a vector of observations (of size \( K \times 1 \)), \( L \) is a linear operator matching the model state \( u(x,t) \) and the observations, \( g_{k} \) define the observational functionals (sampling rules for each datum \( d_k \), i.e., impulses at the observation locations \( x_k \) and times \( t_k \) in our example), and \( e_{d} \) is the observation error.

In (4)–(6), errors \( e, e_{0}, \) and \( e_{d} \) can be interpreted as random processes or vectors, satisfying the equations with
the true state $u$. The optimal, inverse, solution $\tilde{u}$ minimizes the following penalty functional:

$$J(\tilde{u}) = ||\tilde{e}||^2 + ||e_0||_0^2 + ||e_f||_0^2,$$

(7)

where $\tilde{e}$, $e_0$, and $e_f$ are residuals, satisfying (4)–(6) with $u$. The norms in (7) are defined using inverse prior error covariances in the corresponding inputs, $C^{-1}$, $C_0^{-1}$, and $C_f^{-1}$, that must be specified prior to assimilation,

$$||e||^2 = \int_0^T dt \int_S dx_1 \int_0^T dt_2 \int_S dx_2 \ e^T(x_1,t_1)C^{-1}(x_1,t_1,x_2,t_2) \ e(x_2,t_2),$$

(8)

$$||e_0||_0^2 = \int_S dx_1 \int_S dx_2 \ e_0^T(x_1)C_0^{-1}(x_1,x_2) e_0(x_2),$$

(9)

$$||e_f||_0^2 = e_f^T C_f^{-1} e_f.$$  

(10)

The inverse $C^{-1}$ is defined such that

$$\int_0^T dt \int_S dx C(x_1,t_1,x,t)C^{-1}(x,t_1,x_2,t_2) = \delta(x_1-x_2)\delta(t_1-t_2),$$

(11)

where $C(x_1,t_1,x,t) = \langle e(x_1,t_1) e(x,t) \rangle$, and $\langle \rangle$ denotes the ensemble mean [Bennett, 2002]; $C_0^{-1}$ is defined similarly.

[16] In our study, the data error covariance $C_d$ is diagonal with the variances $\sigma_d^2$ on the main diagonal corresponding to the noise level in the data. The choice of the initial and forcing error covariances is less trivial. In models with fine spatial resolution, the assumption that errors are uncorrelated in space and time (e.g., $C = \delta(x_1-x_2)\delta(t_1-t_2)$ and $C_0 = \delta(x_1-x_2)$) could result in an irregular solution that fits the data only in the vicinity of the observational locations [Egbert and Bennett, 1996; Bennett, 2002]. Implementation of the covariances with nonzero spatial and temporal decorrelation scales helps to obtain smooth solutions.

[17] We assume that the errors in $f_{12}$ and $f_{21}$ are statistically independent from each other. For each forcing component, the error covariances are considered in the following general form, allowing nonzero correlation in each spatial dimension and in time, and also cross-shore dependence of the variance:

$$C(x_1,t_1,x_2,t_2) = \gamma^2 \sigma(y_1)\sigma(y_2)C_X(x_1-x_2) \cdot C_Y(y_1-y_2)C_Y(y_1-y_2),$$

(12)

where

$$C_X(x) = \exp\left(-\frac{1 - \cos(qx)}{(q\lambda)^2}\right),$$

(13)

$$C_Y(y) = \exp\left(-\frac{y^2}{(2l_y)^2}\right).$$

(14)

[18] Expression (13), in which $q = 2\pi/L_x$ and $L_x$ is the periodic channel length, provides a functional form that is close to bell-shaped while accounting for periodicity in the alongshore direction [Ménard, 2005]. Numerical values for the scaling factor $\gamma$ and decorrelation scales $l_x$, $l_y$, and $l_z$ utilized in our studies are specified below.

[19] The covariance of (12) is standard for the representor method [Chua and Bennett, 2001]. However, since the true and prior forcings are both steady in our case, it is natural to first consider a special case with $l_z \rightarrow \infty (C = C^\infty)$. Such a covariance does not allow temporal variation in the forcing correction. However, as we shall show, such a severe restriction on the available degrees of freedom in the forcing correction may impede convergence to the true time-variable solution. Using the standard covariance (12) with a finite $l_z$ that is smaller than the characteristic eddy variability timescale ($C = C^{l_z<\infty}$) may result in better convergence, but if rapid variability is allowed in the inverse forcing, the question naturally arises of whether its time-mean will provide an accurate estimate of the constant forcing. In practice, this question is important if one wants to analyze the forcing estimates to obtain inferences about dominant driving mechanisms. In our study, we have been led to consider a composite covariance,

$$C = C^\infty + \alpha C^{l_z<\infty},$$

(16)

where $\alpha$ is a small parameter. With (16), deviations of the inverse forcing from its time-mean are allowed, so there are enough degrees of freedom to allow control of the time-dependent flow, but these deviations are kept small enough to allow accurate estimation of the time-mean forcing.

[20] To define $C_0$, we assume that initial errors in $\zeta$, $u$, and $v$ are statistically independent, with the covariance of each component in the following form:

$$C_0(x_1,x_2) = \gamma_0^2 \sigma_0(y_1)\sigma_0(y_2)C_X(x_1-x_2)C_Y(y_1-y_2).$$

(17)

In our DA experiments, $l_x = 40$ m, $l_y = 20$ m, $\gamma_0 = 0.05$ m, 0.5 m s$^{-1}$, and 0.2 m s$^{-1}$ for initial $\zeta$, $u$, and $v$ respectively. To reduce the error variance in the offshore direction, $\sigma_0(y) = \exp(-y/\gamma)$, with $\gamma = 70$ m. For the forcing error covariance, $\gamma = 0.01$ m$^2$ s$^{-2}$; $l_z = 70$ m, and $l_z = 20$ m. The scaling function $\sigma(y)$ is similar to $\sigma_0$, utilized for the initial conditions, except very close to the coast ($y < 10$ m) where it is linearly reduced to zero. This last modification helps to reduce forcing variations in very shallow water that otherwise may cause numerical instabilities associated with the beach drying ($\zeta < -H$).

3.2. Minimization Method

[22] The representor-based minimization algorithm [Chua and Bennett, 2001] attempts to find a solution to the nonlinear Euler-Lagrange equations, providing necessary conditions for the minimum of (7), iteratively via a series of linearized optimization problems (so called “outer loop” iterations). For details of derivation of the Euler-Lagrange
equations specific to the general form of the model and data defined in (4)–(6) see the Appendix. On iteration \( n + 1 \), the nonlinear equations are linearized with respect to the previous iteration tangent linear inverse solution \( \tilde{u}^n \). On the first outer loop iteration, the system is linearized with respect to \( \tilde{u}^0 = \tilde{u}^\text{prior} (= 0 \) here).

[23] The linearized inverse solution is the sum of the linearized prior (or “forward”) solution \( \tilde{u}^n_p \) and the correction \( \psi^n \),

\[
\tilde{u}^n = \tilde{u}^n_p + \psi^n. 
\]  
(18)

[24] Solution \( \tilde{u}^n_p \) satisfies the full state tangent linear (TL) equations and prior initial conditions,

\[
\frac{\partial \tilde{u}^n_p}{\partial t} = N(\tilde{u}^{n-1}) + A[\tilde{u}^{n-1}] (\tilde{u}^n_p - \tilde{u}^{n-1}) + \tilde{u}^\text{prior},
\]  
(19)

where

\[
A[\tilde{u}^{n-1}] = \frac{\partial N}{\partial u}\bigg|_{u=\tilde{u}^{n-1}},
\]  
(21)

is the TL operator (in our notation, a 3 \times 3 matrix operator with elements depending on \( \tilde{u}^{n-1} \)). Note that in (19), \( N(\tilde{u}^{n-1}) - A[\tilde{u}^{n-1}] \tilde{u}^{n-1} \) provides additional forcing.

[25] The correction \( \psi^n \) satisfies the perturbation TL equations,

\[
\frac{\partial \psi^n}{\partial t} = A[\tilde{u}^{n-1}] \psi^n + \int_S dx_1 \int_0^T dt_1 C(x, t; x_1, t_1) \lambda(x_1, t_1),
\]  
(22)

\[
\psi^n(x, 0) = \int_S dx_1 C_0(x; x_1) \lambda(x_1, 0).
\]  
(23)

[26] The evolution of the sensitivity to inputs \( \lambda(x, t) \) appearing in (22) and (23) is described by the adjoint (AD) system,

\[
\frac{\partial \lambda}{\partial t} - A^t[\tilde{u}^{n-1}] \lambda = [g_1| \ldots | g_K] b = \sum_k g_k b_k,
\]  
(24)

\[
\lambda(x, T) = 0,
\]  
(25)

where

\[
b = C_d^{-1}[d - L(\tilde{u}^n)].
\]  
(26)

[27] Since the inverse solution appears in (26), the TL and AD systems (22)–(25) are coupled. To decouple them, the correction can be written as a linear combination of representer functions \( r_k \), \( k = 1, \ldots, K \). Each representer function describes the spatiotemporal evolution of a correction to \( \tilde{u}^n_p \) that would result from assimilation of a single observation. The representer function \( r_k \) is obtained by solving (22)–(25) with the r.h.s of (24) replaced by \( g_k \) (as a result, \( r_k = \psi^k \)). Then, \( b \) is obtained by solving a \( K \times K \) linear system,

\[
Pb = d - L(\tilde{u}^n),
\]  
(27)

where \( P = R + C_d \) and \( R = L([r_1| r_2| \ldots | r_K]) \) is the representer matrix.

[28] When \( K \) is large, computation and storage of each representer is impractical and unnecessary. Instead, the indirect representer approach is utilized [Egbert et al., 1994; Chua and Bennett, 2001]. System (27) is solved iteratively using the preconditioned conjugate gradient method [Golub and Van Loan, 1989], using a series of “inner loop iterations.” The algorithm requires a rule, by which the matrix \( R \) multiplies an arbitrary vector \( \beta \). This multiplication is achieved by solving (22)–(25) with \( b = \beta \) and sampling the resulting tangent linear solution, to obtain \( L(\psi^n) \). In our implementation, system (27) is solved until \( \varepsilon < 10^{-3} \), where \( \varepsilon \) is the rhs of (27) and \( \varepsilon \) is the misfit in this equation.

[29] In most cases in our study, the linear system (27) is so poorly conditioned that the inner loop iterations fail to converge without preconditioning. Instead of a poorly conditioned system \( Pb = q \), a better conditioned system \( \tilde{P}b = \tilde{q} \), where \( P = P_{pc}^{-1/2} P_p^{1/2} \), \( \tilde{b} = P_{pc}^{-1/2} b \), and \( \tilde{q} = P_{pc}^{-1/2} q \), is solved. Matrix \( P_{pc} \) should be easy to invert since the preconditioned conjugate gradient algorithm requires solving \( Pb = z \), where \( z \) and \( w \) are arbitrary vectors, directly [see Golub and Van Loan, 1989]. The method suggested by Egbert [1997] is found to be very effective. It is based on computation of a small subset of represnters directly and using those to form \( P_{pc} \). Obtaining inversion in a 1-hour interval with \( K = 5,664 \) observations, we use 624 representers to build the preconditioner. On the first outer loop iteration (utilizing the zero solution as the background for linearization), 150–300 inner loop iterations are then required. The second and higher outer loop iterations (utilizing a better background solution) involve 20–50 inner loop iterations. We find that the number of the inner loop iterations depends on the number of degrees of freedom in the corrected inputs. Generally, fewer inner loop iterations are needed if inversion is obtained in a smaller assimilation time interval. Also, in a given time interval, fewer inner loop iterations are needed in a case with \( C = C^\infty \) than with \( C = C^6 \), i.e., when fewer effective degrees of freedom are allowed in the control space.

3.3. Numerical Implementation Details

[30] The TL and AD numerical codes have been developed manually following recipes of adjoint generation [Giering and Kaminski, 1998; Moore et al., 2004]. The background state in both the TL and AD models is provided as a series of snapshots every \( \Delta t \); instantaneous background fields are obtained by linear interpolation. In cases with \( r = 0.004 \) m s\(^{-1} \) (the equilibrated wave regime), we choose \( \Delta t = 1 \) min. With \( r = 0.002 \) m s\(^{-1} \) (a more strongly nonlinear regime), \( \Delta t = 15 \) s to better represent higher-frequency variability. In both the NL and TL models, the discrete space of inputs \( \Phi \) includes the initial conditions and the snapshots of forcing fields provided at selected times; instantaneous forcing at every time step is obtained by linear interpolation. In our implementations, if \( C = C^\infty \),

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the forcing can be represented by two identical snapshots, at \( t = 0 \) and \( T \). In cases allowing time variability in the forcing, forcing snapshots are specified every \( Dt \). The discrete output space \( Y \) includes the snapshots of the fields at times \( t = 0, Dt, 2Dt, \ldots, T \). Copies of the interior fields at the edges of the periodic boundaries are excluded from \( F \) and \( Y \).

Subject to these definitions of the input and output spaces, adjoint symmetry of the TL and AD codes has been maintained to within computer accuracy. Tests of the symmetry involved generating a set of random input vectors \( Q = [f_1 \ldots f_n] \) and verifying that the \( n \times n \) matrix \( Q^T \{AD\} [TL] Q \), where \([TL]\) and \([AD]\) denote the implementation of the TL and AD codes, is symmetric and positive definite.

In the strongly nonlinear regime, small-scale transient flow convergence zones appearing near the coast are aliased in the time series sampled every 15 s. To avoid numerical problems, temporal filtering (using a seven-point polynomial filter) has been applied to the inverse TL solutions before using them as the background state on the next outer loop iteration.

4. Integration Limit Set by Dataless Iterations

Dataless (or “Picard”) iterations are a standard test of convergence of the full state TL model (19) to the NL model [e.g., see Chua and Bennett, 2001; Muccino and Luo, 2005]. On each iteration, (19) is integrated with the true (instead of prior) initial conditions and forcing. On the first iteration, \( \dot{u}_0 = 0 \). Figure 7 shows time series plots of the area-averaged root mean square (RMS) error, i.e., the RMS difference between the true NL solution and TL solutions \( \dot{u}^n \), separately for \( \zeta \), \( u \), and \( v \). The series of the TL solutions converge to the true NL solution only for a limited period of time, \( t_{\text{lim}} \approx 15 \text{ min} \) in the case with \( r = 0.004 \text{ m s}^{-1} \) (Figures 7a–7c) and \( t_{\text{lim}} \approx 7 \text{ min} \) with \( r = 0.002 \text{ m s}^{-1} \) (Figures 7d–7f).

Analysis of the flow fields shows that iteration 1 yields a solution that equilibrates toward a steady sheared current. On iteration 2, shear instabilities appear and grow exponentially on this steady background. Further iterations, utilizing growing solutions as the background in (19), result in the abrupt growth (finite-time blow-up) as the time approaches \( t_{\text{lim}} \). However, the TL solutions apparently converge to the true NL solution for \( t < t_{\text{lim}} \).

Through a series of additional experiments on an alongshore uniform bathymetry (not shown), we verified that \( t_{\text{lim}} \) is proportional to the dominant timescale for growth of the linear shear instabilities on the unstable current profile \( u(y) \). Although \( t_{\text{lim}} \) may indicate a characteristic timescale for the validity of the tangent linearization, it should not stop us from attempting DA over larger time intervals, as discussed next.

5. Data Assimilation Results

The representer method is implemented in a time window of length \( T = 1 \text{ h} \), which is substantially longer than both the limit \( t_{\text{lim}} \) suggested by the dataless iterations, and the dominant timescale of cross-shore variations (several minutes). Our study progresses from the equilibrated wave case (section 5.1) to the more strongly nonlinear case (section 5.2). The quality of DA results are evaluated in terms of several criteria:

1. Do the series of inverse TL solutions converge to a solution that is close to the true nonlinear state? If so, the inverse TL solution provides an accurate dynamically constrained space- and time-continuous interpolation of a sparse data set.

2. Is the inverse estimate of the forcing close to the true forcing? If so, the DA system can qualify as a tool for scientific analysis of dynamical driving mechanisms. We
shall see that success in terms of criterion 1 does not always imply that criterion 2 is met. In fact, the performance will depend on an appropriate choice of $C$.

Do inverse NL solutions, obtained by running the NL model with the inverse initial conditions and forcing on each outer loop iteration, converge to a solution close to the true state? In practice, it may be desirable to obtain a nonlinear solution that fits the data, for example, to facilitate term balance analysis. We shall see that convergence satisfying criterion 1 does not guarantee success in terms of criterion 3.

Approaching these goals, we gain understanding of the following fundamental questions. Is tangent linear growth an obstacle to variational DA over a large time interval, for example, extending far beyond $t_{\text{lim}}$? Can variability in the eddy field, resulting from interior nonlinear flow interactions, be captured in a DA system? What is the impact of the prior forcing error covariance, $C$, both on convergence of the representer method and on estimates of forcing?

5.1. Equilibrated Wave Regime: $r = 4 \times 10^{-3} \text{ m s}^{-1}$

With $C = C^\infty$, the series of inverse TL solutions converges to a solution that fits the data and is uniformly close to the true solution in the full 1-hour time interval, as can be seen in the time series plots of the area-averaged RMS error (Figure 8). Convergence is also apparent in Figure 9 (solid black line), in which the time- and area-averaged $u$- and $v$-RMS errors in $u^*$ are plotted as functions of the outer loop iteration number. The series of inverse NL solutions (half-tone line in Figure 9, iterations 3–5 only) also converges to a solution of the same accuracy as $u^*$. Over the entire assimilation time interval, instantaneous vorticity fields from iteration 5 TL and NL solutions are qualitatively very close to each other, and to the true state. In the equilibrated wave regime, the spatiotemporal evolution of the eddy field can be described deterministically by both the TL and NL inverse solutions.

On iteration 2 in the case with $C = C^\infty$, a sizable, but erroneous, correction is assigned to the alongshore component $f_x$ in a narrow area offshore of the observational array. This correction is positive on average, forcing a strongly sheared alongshore current at $y > 120$ m (Figure 10a), where there are no observations (see Figure 1). Apparently, the DA system attempts to associate small-scale variability in the assimilated fields with the small-scale changes in the time-mean forcing. Additional iterations are necessary to reduce this erroneous effect. In some runs, in which we experimented with the spatial details of $C = C^\infty$, the positive current offshore of the observational array in early iteration solutions was prone to shear instabilities that could not be constrained by DA.

The case with the assumed standard bell-shaped temporal correlation in the forcing error, $C = C^{l<1}$, $l = 300$ s (dashed line in Figure 9), converges more rapidly than
the one with \( C = C^\infty \), even though the assumption \( C = C^\infty \) is actually consistent with the steady forcing used in these experiments. In the case with \( C = C^{\infty} \), the optimization algorithm associates small-scale time-varying variability in observations with the allowed temporal deviations in the forcing, rather than its time mean, resulting in a better (than with \( C = C^\infty \)) solution on iteration 2 (Figure 10b) and faster convergence overall.

[43] A case with the composite covariance (16) was run with \( \alpha = 0.05 \) and \( \ell = 300 \) s. The time- and area-averaged RMS error (dotted line in Figure 9) is close to that with \( C = C^\infty \) for \( u \), and is closer to that with \( C = C^{\infty} \) for \( v \).

[44] For all three covariances considered, the main features of the true forcing (see Figure 2) are reproduced in the inverse estimates after five outer loop iterations (Figures 11, 12, and 13). However, in the alongshore- and time-averaged

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**Figure 11.** Similar to Figure 2, but showing the inverse forcing, \( r = 0.004 \) m s\(^{-1} \), \( C = C^\infty \), iteration 5 solution.

**Figure 12.** Inverse forcing, \( r = 0.004 \) m s\(^{-1} \), \( C = C^{\infty} \), iteration 5 solution, (top) alongshore and (bottom) cross-shore components: (a, c) the alongshore and time averages (lines) ± the standard deviation from the time average (shaded) and (b, d) the time-average fields. In the maps, negative values are shaded. The contour interval is \( 2.5 \times 10^{-3} \) and \( 1 \times 10^{-5} \) m\(^2\) s\(^{-2}\) in Figures 12b and 12d, respectively. The dashed line is the 2-m bathymetric contour.
estimates of \( f_y \) in the case with \( C = C^\infty \) (Figure 11a), the erroneous variability with positive sign in the across-shore direction is still present for \( y > 120 \) m. The standard deviation of \( f_y \) in the case \( C = C^\infty \) is larger than the time-mean in the trough between the coast and the sand bar (see Figure 12a). Utilizing the composite covariance provides additional degrees of freedom compared to the case with the steady covariance and thus helps to avoid the wiggles in the average profile \( f_y(y) \) (see Figure 13a). At the same time, the magnitude of temporal deviations of the forcing is sensibly smaller than in the case using the standard bell-shaped time correlation.

[45] In the equilibrated wave regime, the 1-hour time series solution with \( C = C^\infty \) can be obtained in a way that is more computationally economical than direct minimization in the full 1-hour interval. First, minimization can be performed in a short 5-min time interval beginning on time \( t = 0 \). This converges in three outer loop iterations providing an estimate of the initial conditions and steady forcing that are then used to run the NL model for a longer period. The time series RMS error of that inverse NL solution is shown in Figure 14 as the thick dashed line. Next, optimization is approached in the 15-min interval (also beginning at \( t = 0 \)), using the above-mentioned NL solution as \( \mathbf{u}^0 \). Convergence is now obtained in two outer loop iterations. The improved estimates of the inputs from the 15-min inversion are now

![Figure 13. Similar to Figure 12, but for the case with the composite covariance (16).](image)

![Figure 14. Area-averaged \( v \) RMS error of the prior and nonlinear inverse solutions obtained solving optimization problems sequentially in time windows of increasing length (5, 15, and then 60 min); \( r = 0.004 \) m s\(^{-1} \), \( C = C^\infty \).](image)

![Figure 15. (left) Alongshore and (right) cross-shore components of inverse forcing obtained solving optimization problems sequentially in time windows of increasing length \( T \): (a, b) 5 min, (c, d) 15 min, and (e, f) 60 min; \( r = 0.004 \) m s\(^{-1} \), \( C = C^\infty \). Shades and contours are plotted similar to Figure 2.](image)
used to run the NL model for 1 hour (thin solid line in Figure 14). Finally, this NL solution is used as \( \vec{u}_0 \) during inversion in the 1-hour interval. Only one iteration is needed, and the RMS error of the resulting inverse NL solution (thin dashed line in Figure 14) is close to that of the solution obtained directly in the 1-hour interval.

Comparison of the inverse forcing fields obtained in the 5-, 15-, and 60-min time intervals (Figure 15) to their true values (see Figure 2) suggests that assimilation in the longest interval is essential to obtain an accurate forcing estimate. For instance, in the forcing estimate obtained after 5 min there is no local maximum of \( f_Y \) over the bar (Figure 15b). This feature remains underestimated in the 15-min solution (Figure 15d), but is reproduced accurately in the 1-hour estimate (Figure 15f).

5.2. Irregular Flow Regime: \( r = 0.002 \text{ m s}^{-1} \)

[47] The economical way of obtaining the inverse solution (with \( C = C^{\infty} \)) described at the end of the previous section does not work in the irregular flow regime. Inversion in a 5-min interval converges to a solution that is better than the prior. However, the inverse NL solution resulting from the 15-min inversion does not provide further improvement over the background state.

[48] An attempt to assimilate the data directly in the 1-hour interval with \( C = C^{\infty} \), beginning with \( \vec{u}_0 = 0 \), also fails (Figure 16). Shear instabilities grow exponentially in the iteration 3 TL solution inshore of the bar and the solver fails on iteration 4 owing to numerical instabilities. A similar attempt utilizing \( C = C^{l<\infty} \) with \( l = 300 \text{ s} \) again results in no convergence. However, the case with \( l = 60 \text{ s} \) yields a series of TL inverse solutions converging to a state with a low RMS error. So, allowing degrees of freedom in the forcing correction with temporal scales shorter than 5 min is important. Analysis of the frequency/wave number spectrum of the true solution (see Figures 6e–6h) suggests that motions with frequencies near \( 1/60 = 0.017 \text{ cycles s}^{-1} \) are energetic only inshore of the bar. Adjusting the forcing in the high-frequency part of the spectrum helps to control eddy behavior in this area, where an energetic dipole-shaped vorticity structure with a sharp front travels along the coast (see Figure 4f).

[49] Despite the fact that the inverse TL solution is accurate in the case \( l = 60 \text{ s} \), the corresponding estimate of the forcing is not quite satisfactory (Figure 17). The magnitude of time variations of the alongshore component \( f_x \) is comparable to its time-mean at all \( y \) (Figure 17a). The maximum in \( f_x \) over the bar is shifted to the left (Figure 17b) compared to the true forcing (see Figure 2b). The first maximum of the cross-shore component \( f_y \) (near the coast) is substantially underestimated and the second maximum (over the bar) is absent (Figures 17c and 17d). In an...
application to real data, errors of such magnitude could result in incorrect inferences about the forcing.

To obtain both an accurate estimate of the flow and forcing, the case with the composite covariance, \( t = 60 \text{ s} \) and \( a = 0.2 \) is run. The parameter \( a \) is chosen to be larger than in the weakly nonlinear case, to allow control of more energetic eddy variability. Note that the selected value of \( a \) is close to the ratio of the magnitudes of the time-dependent cross-shore velocity fluctuations and the mean alongshore current. The composite covariance still allows temporal variations on a 1-min scale, but now their magnitude is constrained. Outer loop iterations converge to a TL solution with low RMS error (Figure 18), and the estimate of the forcing (Figure 19) is more accurate than in the previous case, with smaller deviations from the time-mean, better spatial structure in \( f_X \), and two maxima in the cross-shore direction in \( f_Y \).

We also find that the time variable component of the forcing estimate helps to control eddy variability in inverse TL solutions without contributing to the average energy balance. In particular, comparing the time-averaged work by the mean forcing and by its fluctuations, \( |u(f_{X,1} + v(f_{Y,1})| \approx 0.01 \max |\overline{f_X} + \overline{f_Y}| \), where the overbar denotes time-averaging and index 1 the deviation from the time mean. Chua and Bennett [2001] suggest that additional dissipation terms can be introduced in the system to increase the validity limit of the TL model and thus to improve convergence. In our case, the convergence is achieved in the interval substantially longer than \( t_{\text{lim}} \) without introducing additional energy sinks in the system.

We have demonstrated that the solution obtained with the composite covariance meets quality criteria 1 and 2 stated at the beginning of section 5. However, unlike in the equilibrated wave regime, it is more difficult to satisfy criterion 3 in this more irregular regime. The RMS error of the iteration 7 inverse NL solution (Figure 20, half-tone line) is sensibly better than the prior (thick black solid line) for \( \zeta \) and \( u \) (Figures 20a and 20b, respectively) since the time-mean surface elevation set-up \( \zeta \) and alongshore current \( u \) are well predicted by the DA model in this turbulent regime. However, the area-averaged RMS error for the cross-shore velocity component \( v \) (Figure 20c), which would seem to be a good metric of eddy predictability since \( u \approx 0 \), is only marginally better than the prior error. Closer examination of the flow fields shows that in fact the inverse NL solution is qualitatively quite similar to the truth. Figure 21 shows vorticity snapshots every 7.5 min, the true solution on the left and the inverse NL (iteration 7) solution on the right. The two major flow features, the dipole near the coast and a larger-scale eddy farther offshore, are

![Figure 18](image1.png)

**Figure 18.** Time series of area-averaged RMS error, inverse TL solutions, \( r = 0.002 \text{ m s}^{-1} \), composite \( C \): (a) \( \zeta \), (b) \( u \), and (c) \( v \).

![Figure 19](image2.png)

**Figure 19.** Similar to Figure 12, but for the case with \( r = 0.002 \text{ m s}^{-1} \) and the composite \( C \).
reproduced in the inverse NL solution. The location of the latter is well predicted, while the former may be misplaced in the inverse NL solution. The RMS error in $v$ is in fact very sensitive to the location of this dipole vortex structure traveling along the coast.

Note that the corrected initial conditions at $t = 0$ (Figure 21, top left) are overly smooth compared to the true nonlinear solution. Since the time-dependent correction is applied to the forcing, not directly to the ocean state, sharp frontal features are developed in the inverse solutions (both TL and NL) at later times. Preserving sharp fronts would be potentially more difficult if a sequential DA algorithm (e.g., optimal interpolation [Kurapov et al., 2005] or ensemble Kalman filter [Evensen, 1994]) were applied to this problem. Those methods provide corrections to the ocean state sequentially when data are available. Every time the forecast state is corrected, the fronts would be smeared.

Figure 20. Time series of area-averaged RMS error, inverse NL solutions, $r = 0.002 \text{ m s}^{-1}$, composite $C$: (a) $\zeta$, (b) $u$, and (c) $v$.

Figure 21. Vorticity snapshots, the true and inverse NL solutions, $r = 0.002 \text{ m s}^{-1}$, composite $C$, iteration 7. Positive values are shaded, and contours are every $0.01 \text{ s}^{-1}$. 

Spatial maps of the time-averaged RMS velocity error are shown in Figures 22a–22d. In the inverse NL solution, the RMS error of \( u \) is improved throughout the domain compared to the prior RMS error, mostly since the time-mean is better (see Figures 22a and 22c). The RMS error of \( v \) is improved on the offshore side of the bar (Figures 22b and 22d), where the NL model, run with the inverse input estimates, deterministically reproduces spatio-temporal variability in the larger vortex structure. Likewise, the correlation coefficient between the true and inverse NL solutions is above the confidence limit (0.5) offshore of the bar (Figures 22e and 22f). The frequency-wave number power spectra of vorticity in the true and inverse NL solutions (Figure 6, middle and right, respectively), are very similar both inshore and offshore of the bar. So, the variability in the inverse NL solution on eddy timescales appears to be statistically correct throughout the domain.

6. Summary

A key result of our study is that accurate time-continuous reconstruction of the nonlinear flow is possible utilizing variational DA for periods that significantly exceed “the linearity time interval” (as can be defined, for instance, by convergence of Picard iterations). As the DA algorithm converges, the deviation of the inverse solution from the truth may become small enough for the linearization to be valid over large time intervals (1 hour in our nearshore example).

In dynamical regimes where instabilities and eddy interactions drive much of the variability DA is a challenge, particularly if the objective is to improve understanding of forcing mechanisms. Physical instabilities in the dynamics can result in unstable behavior in the inversion procedure. Relaxing tight constraints on the forcing, even though these constraints might be strictly correct, can result in stable inversion and better “control” of unsteady flows in which instabilities and eddy interactions dominate. However, by modifying the covariance to allow fictitious variability in the forcing, the accuracy of the forcing estimate may be compromised. In our example, we utilize the composite covariance (16) to relax the constraint on the forcing (here, steady forcing) only a little, allowing control of the flow with minimal impact on the steady forcing estimates.

In our strongly nonlinear case, variational DA, based on a series of linearized optimization problems, is successful in terms of criteria 1 (convergence of the linearized solutions to the true NL solution) and 2 (improved forcing estimate), defined in the beginning of section 5. In terms of criterion 3 (convergence of NL solutions obtained using optimized initial conditions and forcing), the resulting NL solutions yield an accurate estimate of the mean alongshore current and a eddy field that reproduces qualitatively principal frontal features as well as statistics of the time-
variable flow. It may be argued that the above-mentioned criteria as measures of success are specific to our variational method. With regard to criterion 1, a criticism might be raised that our best solutions satisfy the linearized, rather than nonlinear, dynamical equations and that some other DA methods that do not involve linearized dynamics, for example, ensemble or particle filters, would be a better approach. However, let us note that estimates, or “analyses,” obtained by the filters do not satisfy the nonlinear dynamics either. The sequential fitting of the instantaneous ocean fields to data is analogous to adding a correction term in the dynamical equations [e.g., see Kurapov et al., 2005]. Besides, since the filtering methods do not estimate the forcing directly, their success cannot be evaluated in terms of criteria 2 or 3, used here for the variational approach.

With the steady forcing error covariance $C = C^\infty$, the DA method converges in the equilibrated waves regime, yielding accurate forcing and flow estimates. In this case, an accurate estimate of the steady forcing is necessary to provide correct steady energy input to the system. Given this, the flow meanders over the entire 1-hour interval are largely determined by initial conditions, which define the initial phase of the shear wave. So, the initial conditions (and steady forcing) provide an effective means of control of the time-variable flow. In the irregular case, the flow has limited memory of initial conditions (about 5 min), so the initial conditions are no longer effective means of controlling temporal variability over the 1-hour interval. The representer algorithm with $C = C^\infty$ is not stable in the irregular regime.

Our synthetic data set captures energetic eddy variability in the computational domain. The spatial distribution and areal coverage in our case are comparable, for instance, to an array of moored instruments in the SandyDuck-97 field experiment [Noyes et al., 2004]. Driving the model to close proximity of the true trajectory may be more difficult using sparser data sets. The sensitivity of results to the data array, and more generally studies of convergence and accuracy of variational DA in cases when eddy variability is not adequately sampled, are topics for future research.

In systems with feedback mechanisms between flows and forcings an assumption of time-varying forcing errors would be physically reasonable. For instance, in the nearshore ocean, currents affect wave characteristics modifying the forcing of the circulation [Özkan-Haller and Li, 2003; Yu and Slinn, 2003]. In implementations with real data, analysis of the model residuals could provide insights into these flow-forcing interactions. Conversely, physical understanding of such feedbacks could provide a basis for refining the covariance.

Variational DA in high-resolution models promises to be a useful synthesis tool in the context of oceanographic field programs and emerging ocean observing systems, providing both dynamically constrained interpolation of sparse and diverse data sets, and improved estimates of the forcing and parameters. Our example demonstrates that, for accurate estimation of forcing that varies on timescales much larger than the dominant scales of variability of nonlinear flows, it may be necessary to approach data inversion in time intervals that are substantially longer than the scales associated with eddies (e.g., see discussion at the end of section 5.1). The accuracy of the forcing estimate, and possibly inferences about driving mechanisms, will depend on the choice of the prior forcing error statistics.

The experiment in the alongshore periodic channel is analogous to consideration of the flow in a domain of large alongshore extent. If the data were assimilated in a limited-area model, correcting open boundary inputs could provide additional means of control of the unsteady inverse solution. Effectiveness of such control in the context of unstable nearshore flows remains to be investigated. The complexity of boundary conditions that ensure well-posed mathematical formulation for the shallow-water equations, for instance, nonsmooth dependence on the state [Blayo and Debreu, 2005], present a potential difficulty for the variational method. Resolution of these difficulties in the shallow-water model would be a step toward understanding the even more challenging problems of open boundary control in models of three-dimensional stratified flows [Oliger and Sundström, 1978].

Shear waves, instabilities, and eddies have been observed and modeled in the nearshore surf zone [Noyes et al., 2004, 2005; Long and Özkan-Haller, 2005]. The success of our idealized experiment may open new possibilities for DA in the nearshore ocean. This area can be well-sampled using moored instruments as well as remote sensing [Holman et al., 1993; Chickadel et al., 2003]. DA might be applied to resolve deficiencies of today’s nearshore circulation models, providing improved estimates of the forcing, parameterizations, and possibly bathymetry, as well as improving scientific understanding of the circulation, wave, and morphological processes. Work in that direction will ultimately help develop forecasting capabilities in the nearshore ocean. At the same time, our results have general applicability beyond nearshore applications, for example, to other time-dependent oceanographic flow problems in which the assimilation time interval is long compared to eddy timescales.

Appendix A: Derivation of the Euler-Lagrange Equations

Chua and Bennett [2001] explain the essence of the indirect representer method, including derivation of the TL and AD equations, using a simple one-dimensional advection equation. Following their approach, here we provide details of how to obtain the linearized equations (18)–(27) for the general form of the model and data (4)–(6) and the penalty functional defined by (7)–(10). The variation of the functional near the optimal solution is $\delta J(\vec{u}) = \delta (\vec{u} + \delta \vec{u}) - J(\vec{u})$, where $\delta \vec{u}(x, t)$ is an arbitrary small variation of the state. The “adjoint” variable is defined as the weighted optimum dynamical error estimate,

$$\lambda(x, t) = \int_0^T dt_1 \int_S dx_1 \, C^{-1}(x, t_1, x_1, t_1) \delta e(x_1, t_1).$$

Using (4)–(6) and $N(\vec{u}) + \delta \vec{u} \approx N(\vec{u}) + A(\vec{u})\delta \vec{u}$, $\delta J(\vec{u})$ is expanded, terms quadratic in $\delta \vec{u}$ are dropped and integration by parts is utilized to eliminate $\partial(\delta \vec{u})/\partial t$. Appropriate terms are combined to yield $\delta J(\vec{u}) = \int_S \delta \vec{u}(x, 0) \, q_1 + \int_S \delta \vec{u}(x, T) \, q_2 + \int_S \delta \vec{u}(x, T) \, q_3$. Now, the necessary condition of a minimum of $J$ is $\delta J(\vec{u}) = 0$ for arbitrary $\delta \vec{u}$. Then,
\(q_{i(1,2,3)}\) must be 0, yielding the following Euler-Lagrange equations:

\[
q_i = -\frac{\partial}{\partial t} A'\left[u\right] - |\mathbf{g}_i| + \cdots + |\mathbf{g}_k| C^{-1}_A(d - L(\mathbf{u})) = 0, \tag{A2}
\]

\[
q_i = \lambda(x, t) = 0, \tag{A3}
\]

\[
q_i = \int_{x} C^{-1}_A(x, x) \left(\mathbf{u}(x, 0) - \mathbf{u}_{\text{prior}}\right) dx - \lambda(0, 0) = 0. \tag{A4}
\]

[65] Equations (A2) and (A3) form the “backward” system. The “forward” system is obtained convolving (A1) with \(C_0\) and (A4) with \(C_0\) and using (11),

\[
\frac{\partial \mathbf{u}}{\partial t} = N(\mathbf{u}) + \mathbf{f}_{\text{prior}} + \int_{x}^{T} dt_1 C(x, t, t_1; t_1) \lambda(x, t_1), \tag{A5}
\]

\[
\mathbf{u}(x, 0) = \mathbf{u}_{\text{prior}} + \int_{x} C_0(x, x) \lambda(0, 0) dx. \tag{A6}
\]

[67] Both the optimum state \(\mathbf{u}\) and the adjoint variable \(\lambda\) are unknown such that the backward and forward systems are coupled and nonlinear. They are solved via a series of the outer loop iterations, as described in section 3.2. Linearization is provided with respect to \(\mathbf{u}^{a-1}\). Then, decoupling is done following the representer methodology [see Chua and Bennett, 2001; Bennett, 2002].

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J. S. Allen, G. D. Egbert, A. L. Kurapov, and R. N. Miller, College of Oceanic and Atmospheric Sciences, Oregon State University, 104 COAS Administration Building, Corvallis, OR 97331-5503, USA. (kurapov@coas.oregonstate.edu)